ELSEVIER

Journal of Geometry and Physics 24 (1998) 244–264

JOURNAL OF GEOMETRY AND PHYSICS

Invariants of velocities and higher-order Grassmann bundles ★

Dan Radu Grigore [1], Demeter Krupka *

Department of Mathematics, Silesian University at Opava, Bezrucovo nam. 13, 74601 Opava, Czech Republic

Received 21 February 1996

Abstract

An (r, n)-velocity is an r-jet with source at $0 \in \mathbb{R}^n$, and target in a manifold Y. An (r, n)-velocity is said to be regular if it has a representative which is an immersion at $0 \in \mathbb{R}^n$. The manifold $T_n^r Y$ of (r, n)-velocities as well as its open, L_n^r-invariant, dense submanifold Imm $T_n^r Y$ of regular (r, n)-velocities, are endowed with a natural action of the differential group L_n^r of invertible r-jets with source and target $0 \in \mathbb{R}^n$. In this paper, we describe all continuous, L_n^r-invariant, real-valued functions on $T_n^r Y$ and Imm $T_n^r Y$. We find local bases of L_n^r-invariants Imm $T_n^r Y$ in an explicit, recurrent form. To this purpose, higher-order Grassmann bundles are considered as the corresponding quotients $P_n^r Y = \text{Imm}\, T_n^r Y / L_n^r$, and their basic properties are studied. We show that nontrivial L_n^r-invariants on Imm $T_n^r Y$ cannot be continuously extended onto $T_n^r Y$.

Subj. Class.: Differential invariants; Grassmann bundles
1991 MSC: 53A55, 77S25, 58A20
Keywords: Jet of a mapping; Immersion; Contact element; Differential group; Grassmann bundle; Differential invariant

1. Introduction

By a *velocity* one usually means the derivative of a curve in a smooth manifold Y at a point, or, which is the same, the tangent vector to this curve at a point $y \in Y$. Equivalently,

★ Research of the first author supported by Grant No. 871/95 of the Ministry of Education and Youth of the Czech Republic. Research of the second author supported by Grant No. 201/96/0845 of the Czech Grant Agency.

* Corresponding author. E-mail: demeter.krupka@fpf.slu.cz.

[1] Permanent address: Department of Theoretical Physics, Institute of Atomic Physics, Bucharest-Măgurele, PO Box MG6, România. E-mail: grigoretheor1.ifa.ro, grigoreroifa.ifa.ro.

PII S0393-0440(97)00012-0

such a velocity is a 1-jet with *source* at the origin $0 \in \mathbb{R}$ and *target* in Y. Generalizing this concept one may define an *(r,n)-velocity* as an r-jet with source $0 \in \mathbb{R}^n$ and target in Y. If such an r-jet can be represented by an immersion of a neighborhood of the origin $0 \in \mathbb{R}^n$ into Y, it is called *regular*, and we speak of a *regular (r,n)-velocity*.

Concepts of this kind, i.e., the r-jets of differentiable mappings between smooth manifolds, the contact elements or, which is the same, r-jets of submanifolds, have been introduced in the fiftieth by Ehresmann (see references in [7]), and have become the basic concepts of the theory of differential invariants, and the theory of natural bundles and operators (see [7,10,13,14] and the references there in). These concepts have also been applied in global analysis, and mathematical physics. It should be pointed out, however, that the problem of finding invariants of velocities and the corresponding problem of describing the structure of the space of higher-order velocities has not been touched in the existing monographs on differential invariants and natural bundles [7,13].

The set $T_n^r Y$ of (r,n)-velocities on a smooth manifold is a smooth manifold endowed with a right action of the differential group L_n^r of invertible r-jets with source and target $0 \in \mathbb{R}^n$. The purpose of this paper is to characterize all continuous scalar invariants of this action, i.e. all real-valued functions defined on open subsets of $T_n^r Y$, which are constants on the L_n^r-orbits. Instead of formulating and solving equations for invariant functions we use a different, more powerful method based on considering the quotient space of the open, dense subspace of $T_n^r Y$, formed by regular (r,n)-velocities. The corresponding orbit space is then called the *(r,n)-Grassmann bundle*. It is a fiber bundle over Y whose type fiber is the *(r,n)-Grassmannian*. The canonical quotient projection of the manifold of regular (r,n)-velocities onto the orbit space is the *basis of invariants* of (r,n)-velocities. Geometric structures of this kind as well as their invariants have been studied by M. Krupka [11,12].

Thus, to find all L_n^r-invariants it is enough to find the projection of the manifolds of regular higher-order velocities onto the higher-order Grassmann bundle. We note that an analogous method has been applied to the problem of finding $GL_n(\mathbb{R})$-invariants of a linear connection [9].

Basic concepts of the *first-order* Grassmann bundles has been applied in mathematical physics, and the parametrization independent variational theory (see e.g. [1,3–6]. *Higher-order* Grassmann bundles have become natural underlying structures for the geometric theory of partial differential equations [8].

2. Higher-order velocities

Throughout this paper, $m, n \geq 1$ and $r \geq 0$ are integers such that $n \leq m$, and Y is a smooth manifold of dimension $n+m$.

By an *(r,n)-velocity* at a point $y \in Y$ we mean an r-jet $J_0^r \zeta$ with *source* $0 \in \mathbb{R}^n$ and *target* $y = \zeta(0)$. The set of (r,n)-velocities at y is denoted by $J_{(0,y)}^r(\mathbb{R}^n, Y)$. Further, we denote

$$T_n^r Y = \bigcup_{y \in Y} J_{(0,y)}^r(\mathbb{R}^n, Y),$$

and define surjective mappings $\tau_n^{r,s} : T_n^r Y \to T_n^s Y$, where $0 \le s \le r$, by $\tau_n^{r,s}(J_0^r \zeta) = J_0^s \zeta$. Recall that the set $T_n^r Y$ has a smooth structure defined as follows. Let (V, ψ), $\psi = (y^A)$, be a chart on Y. Then the *associated chart* (V_n^r, ψ_n^r) on $T_n^r Y$ is defined by $V_n^r = (\tau_n^{r,0})^{-1}(V)$, $\psi_n^r = (y^A, y^A_{i_1}, y^A_{i_1 i_2}, \ldots, y^A_{i_1 i_2 \cdots i_r})$, where $1 \le i_1 \le i_2 \le \cdots \le i_r \le n$, and for every $J_0^r \zeta \in V_n^r$,

$$y^A_{i_1 i_2 \cdots i_k}(J_0^r \zeta) = D_{i_1} D_{i_2} \cdots D_{i_k}(y^A \zeta)(0), \quad 0 \le k \le r. \tag{2.1}$$

The set $T_n^r Y$ endowed with the smooth structure defined by the associated charts is called the *manifold of* (r, n)*-velocities* over Y.

The equations of the mapping $\tau_n^{r,s} : T_n^r Y \to T_n^s Y$ in terms of the associated charts are given by $y^A_{i_1 i_2 \cdots i_k} \circ \tau_n^{r,s}(J_0^r \zeta) = y^A_{i_1 i_2 \cdots i_k}(J_0^r \zeta)$, where $0 \le k \le s$. In particular, these mappings are all submersions.

Let tr_t denote the translation $t' \to t' + t$ of $\mathbb{R}^n$. If γ is a smooth mapping of an open set $U \subset \mathbb{R}^n$ into Y, then for any $t \in U$, the mapping $t' \to \gamma \circ \mathrm{tr}_t(t')$ is defined on a neighborhood of the origin $0 \in \mathbb{R}^n$ so that the r-jet $J_0^r(\gamma \circ \mathrm{tr}_t)$ is defined. The mapping

$$U \ni t \to (T_n^r \gamma)(t) = J_0^r(\gamma \circ \mathrm{tr}_t) \in T_n^r Y \tag{2.2}$$

is called the *r-prolongation*, or simply the *prolongation* of γ. Since $y^A_{i_1 i_2 \cdots i_k} \circ T_n^r \gamma(t) = D_{i_1} D_{i_2} \cdots D_{i_k}(y^A(\gamma \circ \mathrm{tr}_t))(0)$ and $D_{i_1}(y^A(\gamma \circ \mathrm{tr}_t))(t') = D_{i_1}(y^A \gamma)(t' + t)$, we get for its chart expression

$$y^A_{i_1 i_2 \cdots i_k} \circ T_n^r \gamma(t) = D_{i_1} D_{i_2} \cdots D_{i_k}(y^A \gamma)(t). \tag{2.3}$$

Assume that we have an element $J_0^r \zeta \in T_n^r Y$. $J_0^r \zeta$ defines the tangent mapping $T_0 T_n^{r-1} \zeta$, which sends a tangent vector $\xi \in T_0 \mathbb{R}^n$ to the tangent vector $T_0 T_n^{r-1} \zeta \cdot \xi$ to $T_n^{r-1} Y$ at $J_0^{r-1} \zeta$. If $\xi = \xi^i (\partial/\partial t^i)_0$, then by (2.3),

$$\begin{aligned} T_0 T_n^{r-1} \zeta \cdot \xi &= \sum_{k=0}^{r-1} \sum_{i_1 \le i_2 \le \cdots \le i_k} \left(\frac{\partial (y^A_{i_1 i_2 \cdots i_k} \circ T_n^{r-1} \zeta)}{\partial t^i} \right)_0 \xi^i \left(\frac{\partial}{\partial y^A_{i_1 i_2 \cdots i_k}} \right)_{J_0^{r-1} \zeta} \\ &= \sum_{k=0}^{r-1} \sum_{i_1 \le i_2 \le \cdots \le i_k} y^A_{i_1 i_2 \cdots i_k i}(J_0^r \zeta) \xi^i \left(\frac{\partial}{\partial y^A_{i_1 i_2 \cdots i_k}} \right)_{J_0^{r-1} \zeta} \\ &= \xi^i d_i(J_0^r \zeta), \end{aligned} \tag{2.4}$$

where

$$d_i = \sum_{k=0}^{r-1} \sum_{i_1 \le i_2 \le \cdots \le i_k} y^A_{i_1 i_2 \cdots i_k i} \frac{\partial}{\partial y^A_{i_1 i_2 \cdots i_k}} \tag{2.5}$$

is a morphism $T_n^r Y \ni J_0^r \zeta \to d_i(J_0^r \zeta) \in T T_n^{r-1} Y$ over $T_n^{r-1} Y$. Indeed, the tangent vectors $d_i(J_0^r \zeta)$ are defined independently of the chosen chart: If $(\overline{V}, \overline{\psi})$, $\overline{\psi} = (\overline{y}^A)$, is some other chart at $\zeta(0)$, then

$$\overline{d}_i = \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} \overline{y}^A_{j_1 j_2 \cdots j_k i} \frac{\partial}{\partial \overline{y}^A_{j_1 j_2 \cdots j_k}},$$

and by (2.4),

$$\overline{d}_i = d_i. \tag{2.6}$$

We note that formula (2.5) does not define a vector field on $T_n^r Y$ since it is not invariant when the tangent vectors $\partial/\partial y^\sigma_{i_1 i_2 \cdots i_k}$ are subject to coordinate transformations on $T_n^r Y$.

Let $f : V_n^{r-1} \to \mathbb{R}$ be a smooth function. We define the ith *formal derivative* $d_i f : V_n^r \to \mathbb{R}$ by

$$d_i f = \sum_{k=0}^{r-1} \sum_{j_1 \le j_2 \le \cdots \le j_k} y^A_{j_1 j_2 \cdots j_k i} \frac{\partial f}{\partial y^A_{j_1 j_2 \cdots j_k}}.$$

By (2.6), the functions $d_i f$ are independent of the charts, and the definition of the ith formal derivative is naturally extended to functions defined on an arbitrary open subset of $T_n^{r-1} Y$.

It can be easily verified that for every smooth function $f : V_n^{r-1} \to \mathbb{R}$ and every smooth mapping γ of an open set $U \subset \mathbb{R}^n$ into Y, $d_i f \circ T_n^r \gamma = D_i (f \circ T_n^{r-1} \gamma)$. In particular, we have for every coordinate function $y^A_{j_1 j_2 \cdots j_k}$,

$$d_i y^A_{j_1 j_2 \cdots j_k} = y^A_{j_1 j_2 \cdots j_k i}. \tag{2.7}$$

Using the formal derivative operators d_i, it is now very easy to find the transformation formulas between two associated charts on $T_n^r Y$. By (2.7) and (2.6),

$$\overline{y}^A_{j_1 j_2 \cdots j_k j_{k+1}} = \overline{d}_{j_{k+1}} \overline{y}^A_{j_1 j_2 \cdots j_k} = d_{j_{k+1}} \overline{y}^A_{j_1 j_2 \cdots j_k} = \cdots = d_{j_{k+1}} \cdots d_{j_2} d_{j_1} \overline{y}^A.$$

This formula may be applied whenever the transformation rules for the coordinate transformations on Y are known.

We shall need a formula for higher-order partial derivatives of the composed mapping in a form well adapted to its use in various inductive calculations in the higher-order differential geometry and the theory of differential invariants.

Let n and k be integers. If $I = \{i_1, i_2, \ldots, i_k\}$ is a set of indices such that $1 \le i_1, i_2, \ldots, i_k \le n$, we usually denote $D_I = D_{i_k} \cdots D_{i_2} D_{i_1}$. Let U and V be open subsets of $\mathbb{R}^n$, let $f : V \to \mathbb{R}$ be a smooth function, and let $\alpha = (\alpha^i)$ be a smooth mapping of U into V. Then one can prove by induction that

$$D_{i_s} \cdots D_{i_2} D_{i_1} (f \circ \alpha)(t) = \sum_{k=1}^{s} \sum_{I=(I_1, I_2, \cdots, I_k)} D_{p_k} \cdots D_{p_2} D_{p_1} f(\alpha(t)) D_{I_k} \alpha^{p_k}(t) \cdots D_{I_2} \alpha^{p_2}(t) D_{I_1} \alpha^{p_1}(t), \tag{2.8}$$

where the second sum is understood to be extended to all partitions $(I_1, I_2, \ldots, I_k)$ of the set $\{i_1, i_2, \ldots, i_s\}$.

Let us write the transformation equations from (V, ψ) to $(\overline{V}, \overline{\psi})$ in the form

$$\overline{y}^A = F^A(y^B). \tag{2.9}$$

We wish to determine explicitly the functions $F^A_{i_1}, F^A_{i_1 i_2}, \dots, F^A_{i_1 i_2 \cdots i_r}$ defining the corresponding transformation

$$\bar{y}^A_{i_1 i_2 \cdots i_k} = F^A_{i_1 i_2 \cdots i_k}(y^B, y^B_{j_1}, y^B_{j_1 j_2}, \dots, y^B_{j_1 j_2 \cdots j_k}), \quad 0 \le k \le r \tag{2.10}$$

from (V^r_n, ψ^r_n) to $(\overline{V}^r_n, \overline{\psi}^r_n)$.

Lemma 1. *The following formula holds:*

$$F^A_{i_1 i_2 \cdots i_s} = \sum_{p=1}^{s} \sum_{(I_1, I_2, \dots, I_p)} y^{B_1}_{I_1} y^{B_2}_{I_2} \cdots y^{B_p}_{I_p} \frac{\partial^p F^A}{\partial y^{B_1} \partial y^{B_2} \cdots \partial y^{B_p}}, \tag{2.11}$$

where the second sum denotes summation over all partitions $(I_1, I_2, \dots, I_p)$ of the set $\{i_1, i_2, \dots, i_s\}$.

Proof. We proceed by induction using (2.7). □

We assume that the reader is familiar with the concept of the differential group. Recall that the *r-th differential group of* $\mathbb{R}^n$, denoted by L^r_n, is the Lie group of invertible r-jets with source and target at $0 \in \mathbb{R}^n$. The group multiplication in L^r_n is defined by the jet composition

$$L^r_n \times L^r_n \ni (J^r_0\alpha, J^r_0\beta) \to J^r_0\alpha \circ J^r_0\beta = J^r_0(\alpha \circ \beta) \in L^r_n, \tag{2.12}$$

where $\circ$ denotes both the composition of mappings, and the composition of r-jets. The *canonical* (global) *coordinates* on L^r_n are defined by

$$a^j_{i_1 i_2 \cdots i_k}(J^r_0\alpha) = D_{i_1} D_{i_2} \cdots D_{i_k} \alpha^j(0), \quad 1 \le k \le r, \ 1 \le i_1 \le i_2 \le \cdots \le i_k \le n, \tag{2.13}$$

where α^j are the components of a representative α of $J^r_0\alpha$.

Lemma 2. *The group multiplication* (2.12) *in L^r_n is expressed in the canonical coordinates* (2.13) *by the equations*

$$c^k_{i_1 i_2 \cdots i_s} = \sum_{p=1}^{s} \sum_{(I_1, I_2, \dots, I_p)} b^{j_1}_{I_1} b^{j_2}_{I_2} \cdots b^{j_p}_{I_p} a^k_{j_1 j_2 \cdots j_p}, \tag{2.14}$$

where $a^k_{i_1 i_2 \cdots i_s} = a^k_{i_1 i_2 \cdots i_s}(J^r_0\alpha)$, $b^k_{i_1 i_2 \cdots i_s} = a^k_{i_1 i_2 \cdots i_s}(J^r_0\beta)$, $c^k_{i_1 i_2 \cdots i_s} = a^k_{i_1 i_2 \cdots i_s}(J^r_0(\alpha \circ \beta))$, and the second sum is extended to all partitions $(I_1, I_2, \dots, I_p)$ of the set $\{i_1, i_2, \dots, i_s\}$.

Proof. We apply (2.8). □

The manifolds of (r, n)-velocities $T^r_n Y$ is endowed with a smooth right action of the differential group L^r_n, defined by the jet composition

$$T^r_n Y \times L^r_n \ni (J^r_0\zeta, J^r_0\alpha) \to J^r_0\zeta \circ J^r_0\alpha = J^r_0(\zeta \circ \alpha) \in T^r_n Y. \tag{2.15}$$

Let us determine the chart expression of this action. To this purpose we use the canonical coordinates a^i_I (2.13) on L^r_n.

Lemma 3. *The group action* (2.15) *is expressed by the equations*

$$\bar{y}^A = y^A, \qquad \bar{y}^A_{i_1 i_2 \cdots i_s} = \sum_{p=1}^{s} \sum_{(I_1, I_2, \cdots, I_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_p}_{I_p} y^A_{j_1 j_2 \cdots j_p}, \tag{2.16}$$

where the second sum is extended to all partitions $(I_1, I_2, \ldots, I_p)$ *of the set* $\{i_1, i_2, \ldots, i_s\}$.

Proof. To prove (2.16), we apply (2.1), (2.15) and (2.8). □

Note the following formula. If γ is a smooth mapping of an open set $U \subset \mathbb{R}^n$ into Y, $U' \subset \mathbb{R}^n$ an open set, and $\alpha : U' \to U$ a smooth mapping, then for every $t \in U'$,

$$T^r_n(\gamma \circ \alpha)(t) = (T^r_n \gamma)(\alpha(t)) \circ J^r_0(\mathrm{tr}_{-\alpha(t)} \circ \alpha \circ \mathrm{tr}_t). \tag{2.17}$$

To derive this formula, we use definition (2.2), and the identity $J^r_0(\gamma \circ \alpha \circ \mathrm{tr}_t) = J^r_0(\gamma \circ \mathrm{tr}_{\alpha(t)}) \circ J^r_0(\mathrm{tr}_{-\alpha(t)} \circ \alpha \circ \mathrm{tr}_t)$. In particular, if α is a diffeomorphism, then $J^r_0(\mathrm{tr}_{-\alpha(t)} \circ \alpha \circ \mathrm{tr}_t) \in L^r_n$, and (2.17) reduces to the group action (2.15).

3. Higher-order Grassmann bundles

An (r, n)-velocity $J^r_0 \zeta \in T^r_n Y$ is said to be *regular* if it has a representative which is an immersion. If (V, ψ), $\psi = (y^A)$, is a chart, and the target $\zeta(0)$ of an element $J^r_0 \zeta \in T^r_n Y$ belongs to V, then $J^r_0 \zeta$ is regular if and only if there exists a subsequence $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ of the sequence $(1, 2, \ldots, n, n+1, \ldots, n+m)$ such that $\det D_i(y^{\nu_k} \circ \zeta)(0) \neq 0$. Regular (r, n)-velocities form an open, L^r_n-invariant subset of $T^r_n Y$, which is called the *manifold of regular* (r, n)*-velocities*, and is denoted by $\mathrm{Imm}\, T^r_n Y$. Recall that $\mathrm{Imm}\, T^r_n Y$ is endowed with a smooth right action of the differential group L^r_n, defined by restricting (2.15), i.e. by

$$\mathrm{Imm}\, T^r_n Y \times L^r_n \ni (J^r_0 \zeta, J^r_0 \alpha) \to J^r_0 \zeta \circ J^r_0 \alpha = J^r_0(\zeta \circ \alpha) \in \mathrm{Imm}\, T^r_n Y. \tag{3.1}$$

If $a^k_{i_1}, a^k_{i_1 i_2}, \ldots, a^k_{i_1 i_2 \ldots i_r}$ are the canonical coordinates on L^r_n, this action is expressed by the equations

$$\bar{y}^A = y^A, \qquad \bar{y}^A_{i_1 i_2 \cdots i_s} = \sum_{p=1}^{s} \sum_{(I_1, I_2, \ldots, I_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_p}_{I_p} y^A_{j_1 j_2 \cdots j_p}. \tag{3.2}$$

In the proof of the following result we construct, among others, a complete system of L^r_n-invariants of the action (3.1) on $\mathrm{Imm}\, T^r_n Y$. We use the *associated charts* on $\mathrm{Imm}\, T^r_n Y$, which are defined as intersections of $\mathrm{Imm}\, T^r_n Y$ with associated charts on $T^r_n Y$.

Theorem 1. *The group action* (3.1) *defines on* $\mathrm{Imm}\, T^r_n Y$ *the structure of a right principal* L^r_n*-bundle.*

Proof. We have to show that (a) the equivalence $\mathcal{R}$ "there exists $J_0^r\alpha \in L_n^r$ such that $J_0^r\zeta = J_0^r\chi \circ J_0^r\alpha$" is a closed submanifold of the product manifold $\operatorname{Imm} T_n^r Y \times \operatorname{Imm} T_n^r Y$, and (b) the group action (3.1) is free.

(a) First we construct an atlas on $\operatorname{Imm} T_n^r Y$, adapted to the group action (3.1).

Let (V, ψ), $\psi = (y^A)$, be a chart on Y, (V_n^r, ψ_n^r), $\psi_n^r = (y^A, y_{j_1}^A, \dots, y_{j_1 j_2 \cdots j_r}^A)$, the associated chart on $\operatorname{Imm} T_n^r Y$. We set for every subsequence $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ of the sequence $(1, 2, \dots, n, n+1, \dots, n+m)$

$$W^\nu = \{J_0^r\zeta \in V_n^r \mid \det(y_j^{\nu_k}(J_0^r\zeta) \neq 0\}. \tag{3.3}$$

W^ν is an open, L_n^r-invariant subset of V_n^r, and

$$\bigcup_\nu W^\nu = V_n^r.$$

Restricting the mapping ψ_n^r to W^ν we obtain a chart (W^ν, ψ_n^r).

The equivalence $\mathcal{R}$ is obviously covered by the open sets of the form $W^\nu \times W^\nu$. We shall find its equations in terms of the charts $(W^\nu \times W^\nu, \psi_n^r \times \psi_n^r)$. Let us consider the set $\mathcal{R} \cap (\mathcal{W}^\nu \times \mathcal{W}^\nu)$. Assume for simplicity that $\nu = (1, 2, \dots, n)$. A point $(J_0^r\zeta, J_0^r\chi) \in W^\nu \times W^\nu$ belongs to $\mathcal{R} \cap (\mathcal{W}^\nu \times \mathcal{W}^\nu)$ if and only if there exists $J_0^r\alpha \in L_n^r$ such that $J_0^r\zeta = J_0^r\chi \circ J_0^r\alpha$ or, which is the same, if and only if the system of equations (3.2) has a solution $a_{i_1}^k, a_{i_1 i_2}^k, \dots, a_{i_1 i_2 \cdots i_r}^k$. Clearly, in this system $\bar{y}^A, \bar{y}_{i_1}^A, \bar{y}_{i_1 i_2}^A, \dots, \bar{y}_{i_1 i_2 \cdots i_r}^A$ (resp. $y^A, y_{p_1}^A, y_{p_1 p_2}^A, \dots, y_{p_1 p_2 \cdots p_r}^A$) are coordinates of $J_0^r\zeta$ (resp. $J_0^r\chi$). But on W^ν, $\det(y_i^k) \neq 0$, where $1 \le i, k \le n$. Consequently, there exist functions $z_j^i : W^\nu \to \mathbb{R}$ such that $z_j^i y_i^k = \delta_j^k$. Conditions (3.2) now imply, for $A = k = 1, 2, \dots, n$,

$$\begin{aligned}\bar{y}_{i_1 i_2 \cdots i_s}^k &= \sum_{p=1}^{s} \sum_{(I_1, I_2, \dots, I_p)} a_{I_1}^{j_1} a_{I_2}^{j_2} \cdots a_{I_p}^{j_p} y_{j_1 j_2 \cdots j_p}^k \\ &= \sum_{p=2}^{s} \sum_{(I_1, I_2, \dots, I_p)} a_{I_1}^{j_1} a_{I_2}^{j_2} \cdots a_{I_p}^{j_p} y_{j_1 j_2 \cdots j_p}^k + a_{i_1 i_2 \cdots i_s}^{j_1} y_{j_1}^k,\end{aligned}$$

which allows us to determine the canonical coordinates of the group element $J_0^r\alpha$ by the recurrent formula

$$a_{i_1 i_2 \cdots i_s}^q = z_k^q \left(\bar{y}_{i_1 i_2 \cdots i_s}^k - \sum_{p=2}^{s} \sum_{(I_1, I_2, \dots, I_p)} a_{I_1}^{j_1} a_{I_2}^{j_2} \cdots a_{I_p}^{j_p} y_{j_1 j_2 \cdots j_p}^k \right). \tag{3.4}$$

Taking $A = \sigma = n+1, n+2, \dots, n+m$ in (3.2) and substituting from (3.4) we get

$$\bar{y}_{i_1 i_2 \cdots i_s}^\sigma = \sum_{p=1}^{s} \sum_{(I_1, I_2, \dots, I_p)} a_{I_1}^{j_1} a_{I_2}^{j_2} \cdots a_{I_p}^{j_p} y_{j_1 j_2 \cdots j_p}^\sigma, \tag{3.5}$$

where the group parameters a_I^j are all certain rational functions of $y_{j_1 j_2 \cdots j_s}^\lambda, \bar{y}_{j_1 j_2 \cdots j_s}^\lambda$. These are the desired equations of the equivalence $\mathcal{R}$ on $W^\nu \times W^\nu$.

Now define a new chart on $\mathrm{Imm}\, T_n^r Y \times \mathrm{Imm}\, T_n^r Y$, $(W^\nu \times W^\nu, \Phi^\nu)$, where $\Phi^\nu = (y^A, y^A_{j_1}, y^A_{j_1 j_2}, \ldots, y^A_{j_1 j_2 \cdots j_r}, \Phi^\sigma, \Phi^\sigma_{j_1}, \Phi^\sigma_{j_1 j_2}, \ldots, \Phi^\sigma_{j_1 j_2 \cdots j_r}, \bar{y}^k, \bar{y}^k_{j_1}, \bar{y}^k_{j_1 j_2}, \ldots, \bar{y}^k_{j_1 j_2 \cdots j_r})$ is the collection of coordinate functions, by

$$\Phi^\sigma = \bar{y}^\sigma - y^\sigma, \qquad \Phi^\sigma_{i_1 i_2 \cdots i_s} = \bar{y}^\sigma_{i_1 i_2 \cdots i_s} - \sum_{p=1}^{s} \sum_{(I_1, I_2, \ldots, I_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_p}_{I_p} y^\sigma_{j_1 j_2 \cdots j_p}.$$

In terms of this new chart, the equivalence $\mathcal{R}$ has equations $\Phi^\sigma = 0$, $\Phi^\sigma_{i_1 i_2 \cdots i_s} = 0$, and is therefore a closed submanifold of $\mathrm{Imm}\, T_n^r Y \times \mathrm{Imm}\, T_n^r Y$.

(b) Assume that for some $J_0^r \zeta \in \mathrm{Imm}\, T_n^r Y$ and $J_0^r \alpha \in L_n^r$, $J_0^r \zeta \circ J_0^r \alpha = J_0^r \zeta$. Then Eqs. (3.2) reduce to

$$y^A_{i_1 i_2 \cdots i_s} = \sum_{p=1}^{s} \sum_{(I_1, I_2, \ldots, I_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_p}_{I_p} y^A_{j_1 j_2 \cdots j_p},$$

which gives us, using (3.4), $a_i^p = \delta_i^p$, $a^p_{i_1 i_2} = 0, \ldots, a^p_{i_1 i_2 \cdots i_r} = 0$, i.e., $J_0^r \alpha = J_0^r \mathrm{id}_{\mathbb{R}^n}$.

This completes the proof. □

We have the following corollaries.

Corollary 1. *The orbit space $P_n^r Y = \mathrm{Imm}\, T_n^r Y / L_n^r$ has a unique smooth structure such that the canonical quotient projection $\rho_n^r : \mathrm{Imm}\, T_n^r Y \to P_n^r Y$ is a submersion. The dimension of $P_n^r Y$ is*

$$\dim P_n^r Y = m \binom{n+r}{n} + n.$$

The following corollary solves the problem of finding all L_n^r-invariant functions on $\mathrm{Imm}\, T_n^r Y$. It says that the projection $\rho_n^r : \mathrm{Imm}\, T_n^r Y \to P_n^r Y$ is the basis of L_n^r-invariant functions.

Corollary 2. *Every L_n^r-invariant function $f : W \to \mathbb{R}$, where $W \subset \mathrm{Imm}\, T_n^r Y$ is an L_n^r-invariant open set, can be factored through the projection mapping $\rho_n^r : \mathrm{Imm}\, T_n^r Y \to P_n^r Y$.*

Now we are going to construct charts on $\mathrm{Imm}\, T_n^r Y$ adapted to the right action (3.1) of the differential group L_n^r. We may consider, for example, the charts (3.3) with $\nu = (1, 2, \ldots, n)$.

Theorem 2. *Let (V, ψ), $\psi = (y^A)$, be a chart on Y, (V_n^r, ψ_n^r), $\psi_n^r = (y^A_{i_1 i_2 \cdots i_s})$, $s \le r$, the associated chart on $\mathrm{Imm}\, T_n^r Y$, and $W = \{J_0^r \zeta \in V_n^r \mid \det(y_j^i(J_0^r \zeta)) \neq 0\}$, $1 \le i, j \le n$. There exist unique functions $w^\sigma, w^\sigma_{j_1}, w^\sigma_{j_1 j_2}, \ldots, w^\sigma_{j_1 j_2 \cdots j_r}$ defined on W such that*

$$y^\sigma = w^\sigma, \qquad y^\sigma_{p_1 p_2 \cdots p_k} = \sum_{q=1}^{k} \sum_{(I_1, I_2, \ldots, I_q)} y^{j_1}_{I_1} y^{j_2}_{I_2} \cdots y^{j_q}_{I_q} w^\sigma_{j_1 j_2 \cdots j_q}. \tag{3.6}$$

The pair (W, Φ), *where* $\Phi = (w^\sigma, w^\sigma_{p_1}, w^\sigma_{p_1 p_2}, \ldots, w^\sigma_{p_1 p_2 \ldots p_r}, y^i, y^i_{j_1}, y^i_{j_1 j_2}, \ldots, y^i_{j_1 j_2 \cdots j_r})$, *is a chart on* Imm $T^r_n Y$. *The functions* $w^\sigma, w^\sigma_{j_1}, w^\sigma_{j_1 j_2}, \ldots, w^\sigma_{j_1 j_2 \cdots j_r}$ *satisfy the recurrent formula*

$$w^\sigma_{j_1 j_2 \cdots j_k j_{k+1}} = z^s_{j_{k+1}} d_s w^\sigma_{j_1 j_2 \cdots j_k}, \tag{3.7}$$

and are L^r_n*-invariant.*

Proof. We proceed by induction.

(1) We prove that the assertion is true for $r = 1$. Consider the pair (W, Φ), $\Phi = (w^\sigma, w^\sigma_{p_1}, y^i, y^i_{j_1})$, where $w^\sigma = y^\sigma$, $w^\sigma_j = z^k_j y^\sigma_k$. Obviously $y^\sigma_p = y^j_p w^\sigma_j$, which implies that (W, Φ) is a new chart. Moreover, $w^\sigma_j = z^k_j d_k y^\sigma = z^k_j d_k w^\sigma$. It remains to show that the functions w^σ_j are L^1_n-invariant. Since the group action (3.2) is expressed by $\bar{y}^i = y^i$, $\bar{y}^\sigma = y^\sigma$, $\bar{y}^i_p = a^j_p y^i_j$, $\bar{y}^\sigma_p = a^j_p y^\sigma_j$, the inverse of the matrix $\bar{y}^i_p = a^j_p y^i_j$ is $\bar{z}^p_q = z^s_q b^p_s$, where b^p_s stands for the inverse of a^p_s. Hence $\bar{w}^\sigma_j = \bar{z}^k_j \bar{y}^\sigma_k = z^s_j b^k_s a^p_k y^\sigma_p = z^p_j y^\sigma_p = w^\sigma_j$ proving the invariance.

(2) Assume that formulas (3.6), (3.7) hold for $k = r - 1$. Write (3.6) in the form

$$y^\sigma_{p_1 p_2 \cdots p_k} = \sum_{q=1}^{k} \sum_{(I_1, I_2, \ldots, I_q)} y^{j_1}_{I_1} y^{j_2}_{I_2} \cdots y^{j_q}_{I_q} w^\sigma_{j_1 j_2 \cdots j_q}.$$

Then

$$\begin{aligned} y^\sigma_{p_1 p_2 \cdots p_k p_{k+1}} &= d_{p_{k+1}} y^\sigma_{p_1 p_2 \cdots p_k} \\ &= \sum_{q=1}^{k} \sum_{(I_1, I_2, \ldots, I_q)} (d_{p_{k+1}} (y^{j_1}_{I_1} y^{j_2}_{I_2} \cdots y^{j_q}_{I_q}) w^\sigma_{j_1 j_2 \cdots j_q} \\ &\qquad + y^{j_1}_{I_1} y^{j_2}_{I_2} \cdots y^{j_q}_{I_q} y^{j_{q+1}}_{p_{k+1}} z^s_{j_{q+1}} d_s w^\sigma_{j_1 j_2 \cdots j_q}). \end{aligned}$$

In this formula we sum through all partitions $(I_1, I_2, \ldots, I_q)$ of the set $\{p_1, p_2, \ldots, p_k\}$. On the other hand, when passing to all partitions $(J_1, J_2, \ldots, J_q)$ of the set $\{p_1, p_2, \ldots, p_k, p_{k+1}\}$ we get

$$\begin{aligned} & y^\sigma_{p_1 p_2 \ldots p_k p_{k+1}} \\ &= \sum_{q=1}^{k} \sum_{(I_1, I_2, \ldots, I_q)} (d_{p_{k+1}} (y^{j_1}_{I_1} y^{j_2}_{I_2} \cdots y^{j_q}_{I_q}) w^\sigma_{j_1 j_2 \cdots j_q} \\ &\qquad + y^{j_1}_{I_1} y^{j_2}_{I_2} \cdots y^{j_q}_{I_q} y^{j_{q+1}}_{p_{k+1}} z^s_{j_{q+1}} d_s w^\sigma_{j_1 j_2 \cdots j_q}) \\ &= \sum_{q=1}^{k} \sum_{(J_1, J_2, \ldots, J_q)} y^{j_1}_{J_1} y^{j_2}_{J_2} \cdots y^{j_q}_{J_q} w^\sigma_{j_1 j_2 \cdots j_q} + y^{j_1}_{p_1} y^{j_2}_{p_2} \cdots y^{j_q}_{p_q} y^{j_{q+1}}_{p_{k+1}} z^s_{j_{q+1}} d_s w^\sigma_{j_1 j_2 \cdots j_q}, \end{aligned} \tag{3.8}$$

and we see that (3.8) has the same form as (3.6), where $w^{\sigma}_{j_1 j_2 \cdots j_k j_{k+1}} = z^{s}_{j_{k+1}} d_s w^{\sigma}_{j_1 j_2 \cdots j_k}$. Uniqueness is immediate since $w^{\sigma}_{j_1 j_2 \cdots j_k j_{k+1}}$ may be expressed explicitly from (3.8).

It remains to prove the invariance condition $\bar{w}^{\sigma}_{j_1 j_2 \cdots j_s} = w^{\sigma}_{j_1 j_2 \cdots j_s}$.

Since the points $J^r_0 \zeta \circ J^r_0 \alpha$ and $J^r_0 \zeta$ belong to the same orbit, their coordinates satisfy the recurrence formula (3.5):

$$\bar{y}^{\sigma}_{i_1 i_2 \cdots i_s} = \sum_{p=1}^{s} \sum_{(I_1, I_2, \ldots, I_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_p}_{I_p} y^{\sigma}_{j_1 j_2 \cdots j_p}, \quad s = 1, 2, \ldots, r,$$

in which

$$a^{q}_{k_1 k_2 \cdots k_t} = z^{q}_{k} \left(\bar{y}^{k}_{i_1 i_2 \cdots i_t} - \sum_{p=2}^{t} \sum_{(K_1, K_2, \ldots, K_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_p}_{I_p} y^{k}_{j_1 j_2 \cdots j_p} \right)$$

for all $t \le s$ (see (3.4)). Here $(I_1, I_2, \ldots I_p)$ is a partition of the set $\{i_1, i_2, \ldots i_s\}$ and $(K_1, K_2, \ldots, K_p)$ is a partition of the set $\{k_1, k_2, \ldots, k_t\}$. Using (3.6) we can write

$$\bar{y}^{\sigma}_{i_1 i_2 \cdots i_s} = \sum_{p=1}^{s} \sum_{(I_1, I_2, \ldots, I_p)} \bar{y}^{j_1}_{I_1} \bar{y}^{j_2}_{I_2} \cdots \bar{y}^{j_p}_{I_p} \bar{w}^{\sigma}_{j_1 j_2 \ldots j_p},$$

$$y^{\sigma}_{j_1 j_2 \cdots j_s} = \sum_{l=1}^{p} \sum_{(J_1, J_2, \ldots, J_l)} y^{t_1}_{J_1} y^{t_2}_{J_2} \cdots y^{t_l}_{J_l} w^{\sigma}_{t_1 t_2 \cdots t_l},$$

where $(I_1, I_2, \ldots, I_p)$ is a partition of the set $\{i_1, i_2, \ldots, i_s\}$ and $(J_1, J_2, \ldots, J_l)$ is a partition of the set $\{j_1, j_2, \ldots, j_p\}$. This gives us the equation

$$\sum_{p=1}^{s} \sum_{(I_1, I_2, \ldots, I_p)} \bar{y}^{j_1}_{I_1} \bar{y}^{j_2}_{I_2} \cdots \bar{y}^{j_p}_{I_p} \bar{w}^{\sigma}_{j_1 j_2 \cdots j_p}$$
$$= \sum_{p=1}^{s} \sum_{(I_1, I_2, \ldots, I_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_p}_{I_p} \left(\sum_{l=1}^{p} \sum_{(J_1, J_2, \ldots, J_l)} y^{t_1}_{J_1} y^{t_2}_{J_2} \cdots y^{t_l}_{J_l} w^{\sigma}_{t_1 t_2 \cdots t_l} \right). \tag{3.9}$$

Now we wish to determine the terms $w^{\sigma}_{t_1 t_2 \cdots t_p}$ on the right-hand side with fixed p. Changing the notation of the indices, we get the expression

$$\sum_{q=1}^{s} \sum_{(I_1, I_2, \ldots, I_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_q}_{I_q} \left(\sum_{p=1}^{q} \sum_{(J_1, J_2, \ldots, J_p)} y^{t_1}_{J_1} y^{t_2}_{J_2} \cdots y^{t_p}_{J_p} w^{\sigma}_{t_1 t_2 \cdots t_p} \right)$$

from which we see that $w^{\sigma}_{t_1 t_2 \cdots t_p}$ are contained in every summand with $q \ge p$. Thus, the required terms are given by

$$\left(\sum_{q=p}^{s} \sum_{(I_1, I_2, \ldots, I_q)} \sum_{(J_1, J_2, \ldots, J_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_q}_{I_q} y^{t_1}_{J_1} y^{t_2}_{J_2} \cdots y^{t_p}_{J_p} \right) w^{\sigma}_{t_1 t_2 \cdots t_p}.$$

In this formula $(I_1, I_2, \ldots, I_q)$ is a partition of the set $\{i_1, i_2, \ldots, i_s\}$, and $(J_1, J_2, \ldots, J_p)$ is a partition of the set $\{j_1, j_2, \ldots, j_q\}$.

Now we adopt the following notation. If $I = (i_1, i_2, \ldots, i_s)$ is a multi-index, then $(I_1, I_2, \ldots, I_p) \sim I$ means that $(I_1, I_2, \ldots, I_p)$ is a partition of the set $\{i_1, i_2, \ldots, i_s\}$.

As before, let $I = (i_1, i_2, \ldots, i_s)$, and let p be fixed. We wish to show that

$$\left(\sum_{(I_1, I_2, \ldots, I_p)} \bar{y}^{j_1}_{I_1} \bar{y}^{j_2}_{I_2} \cdots \bar{y}^{j_p}_{I_p}\right) w^{\sigma}_{j_1 j_2 \cdots j_p}$$
$$= \left(\sum_{q=p}^{s} \sum_{(I_1, I_2, \ldots, I_q)} \sum_{(J_1, J_2, \ldots, J_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_q}_{I_q} y^{t_1}_{J_1} y^{t_2}_{J_2} \cdots y^{t_p}_{J_p}\right) w^{\sigma}_{t_1 t_2 \cdots t_p}. \tag{3.10}$$

Write the transformation formula (2.16) in the form

$$\bar{y}^A_I = \sum_{p=1}^{|I|} \sum_{(I_1, I_2, \ldots, I_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_p}_{I_p} y^A_{j_1 j_2 \cdots j_p}, \quad (I_1, I_2, \ldots, I_p) \sim I.$$

Using the same notation, we have

$$\bar{y}^{t_k}_{I_k} = \sum_{q_k=1}^{|I_k|} \sum_{(I_{k,1}, I_{k,2}, \ldots, I_{k,q_k})} a^{j_{k,1}}_{I_{k,1}} a^{j_{k,2}}_{I_{k,2}} \cdots a^{j_{k,q_k}}_{I_{k,q_k}} y^{t_k}_{j_{k,1} j_{k,2} \cdots j_{k,q_k}},$$
$$(I_{k,1}, I_{k,2}, \ldots, I_{k,q_k}) \sim I_k,$$

where $(I_1, I_2, \ldots, I_p) \sim I$. Thus,

$$\left(\sum_{(I_1, I_2, \ldots, I_p)} \bar{y}^{t_1}_{I_1} \bar{y}^{t_2}_{I_2} \cdots \bar{y}^{t_p}_{I_p}\right) w^{\sigma}_{t_1 t_2 \cdots t_p}$$
$$= \left(\sum_{q_1=1}^{|I_1|} \sum_{(I_{1,1}, I_{1,2}, \ldots, I_{1,q_1})} a^{j_{1,1}}_{I_{1,1}} a^{j_{1,2}}_{I_{1,2}} \cdots a^{j_{1,q_1}}_{I_{1,q_1}} y^{t_1}_{J_1}\right)$$
$$\times \left(\sum_{q_2=1}^{|I_2|} \sum_{(I_{2,1}, I_{2,2}, \ldots, I_{2,q_2})} a^{j_{2,1}}_{I_{2,1}} a^{j_{2,2}}_{I_{2,2}} \cdots a^{j_{2,q_2}}_{I_{2,q_2}} y^{t_2}_{J_2}\right)$$
$$\times \cdots \left(\sum_{q_p=1}^{|I_p|} \sum_{(I_{p,1}, I_{p,2}, \ldots, I_{p,q_p})} a^{j_{p,1}}_{I_{p,1}} a^{j_{p,2}}_{I_{p,2}} \cdots a^{j_{p,q_p}}_{I_{p,q_p}} y^{t_p}_{J_p}\right) w^{\sigma}_{t_1 t_2 \cdots t_p},$$

where $J_1 = (j_{1,1}, j_{1,2}, \ldots, j_{1,q_1})$, $J_2 = (j_{2,1}, j_{2,2}, \ldots, j_{2,q_2})$, ... and $J_p = (j_{p,1}, j_{p,2}, \ldots, j_{p,q_p})$.

This expression can be written in a different way. Notice that since $(I_{i,1}, I_{i,2}, \ldots, I_{i,q_i}) \sim I_i$, then

$$(I_{1,1}, I_{1,2}, \ldots, I_{1,q_1}, I_{2,1}, I_{2,2}, \ldots, I_{2,q_2}, \ldots, I_{p,1}, I_{p,2}, \ldots, I_{p,q_p}) \sim I,$$
$$|I_1| + |I_2| + \cdots + |I_p| = |I| = s,$$

and if we define $q = q_1 + q_2 + \cdots + q_p$, we get $p \leq q \leq |I_1| + |I_2| + \cdots + |I_p| = |I| = s$. Now, having in mind the corresponding summation ranges,

$$\left(\sum_{(I_1, I_2, \ldots, I_p)} \bar{y}^{t_1}_{I_1} \bar{y}^{t_2}_{I_2} \cdots \bar{y}^{t_p}_{I_p} \right) w^{\sigma}_{t_1 t_2 \cdots t_p}$$
$$= \sum a^{j_{1,1}}_{I_{1,1}} a^{j_{1,2}}_{I_{1,2}} \cdots a^{j_{1,q_1}}_{I_{1,q_1}} a^{j_{2,1}}_{I_{2,1}} a^{j_{2,2}}_{I_{2,2}} \cdots a^{j_{2,q_2}}_{I_{2,q_2}} \cdots a^{j_{p,1}}_{I_{p,1}} a^{j_{p,2}}_{I_{p,2}}$$
$$\times \cdots a^{j_{p,q_p}}_{I_{p,q_p}} y^{t_1}_{J_1} y^{t_2}_{J_2} \cdots y^{t_p}_{J_p} w^{\sigma}_{t_1 t_2 \cdots t_p}.$$

If we denote

$$(s_1, s_2, \ldots, s_q)$$
$$= (j_{1,1}, j_{1,2}, \ldots, j_{1,q_1}, j_{2,1}, j_{2,2}, \ldots, j_{2,q_2}, \ldots, j_{p,1}, j_{p,2}, \ldots, j_{p,q_p})$$

and

$$(P_1, P_2, \ldots, P_q)$$
$$= (I_{1,1}, I_{1,2}, \ldots, I_{1,q_1}, I_{2,1}, I_{2,2}, \ldots, I_{2,q_2}, \ldots, I_{p,1}, I_{p,2}, \ldots, I_{p,q_p}),$$

it is immediate that $(P_1, P_2, \ldots, P_q) \sim I$, and

$$\left(\sum_{(I_1, I_2, \ldots, I_p)} \bar{Y}^{J_1}_{I_1} \bar{y}^{j_2}_{I_2} \cdots \bar{y}^{j_p}_{I_p} \right) w^{\sigma}_{j_1 j_2 \cdots j_p}$$
$$= \left(\sum_{q=p}^{s} \sum_{(P_1, P_2, \ldots, P_q)} \sum_{(J_1, J_2, \ldots, J_p)} a^{s_1}_{P_1} a^{s_2}_{P_2} \cdots a^{s_q}_{P_q} y^{t_1}_{J_1} y^{t_2}_{J_2} \cdots y^{t_p}_{J_p} \right) w^{\sigma}_{t_1 t_2 \cdots t_p}.$$

This proves (3.10).

Returning to (3.9), and substituting from (3.10) we get a basic formula

$$\sum_{p=1}^{s} \sum_{(I_1, I_2, \ldots, I_p)} \bar{y}^{j_1}_{I_1} \bar{y}^{j_2}_{I_2} \cdots \bar{y}^{j_p}_{I_p} (\bar{w}^{\sigma}_{j_1 j_2 \cdots j_p} - w^{\sigma}_{j_1 j_2 \cdots j_p}) = 0. \tag{3.11}$$

Now it is easy to show that $\bar{w}^{\sigma}_{j_1 j_2 \cdots j_s} = w^{\sigma}_{j_1 j_2 \cdots j_s}$ provided $\bar{w}^{\sigma}_{j_1 j_2 \cdots j_k} = w^{\sigma}_{j_1 j_2 \cdots j_k}$ for all $k \leq s - 1$.

If $s = 1$, we get $\bar{y}^{j_1}_{i_1} (\bar{w}^{\sigma}_{j_1} - w^{\sigma}_{j_1}) = 0$, and since the matrix $\bar{y}^{j}_{i}$ is regular, $\bar{w}^{\sigma}_{j} = w^{\sigma}_{j}$.

If $s = 2$, we have $\bar{y}^{j_1}_{i_1 i_2} (\bar{w}^{\sigma}_{j_1} - w^{\sigma}_{j_1}) + \bar{y}^{j_1}_{i_1} \bar{y}^{j_2}_{i_2} (\bar{w}^{\sigma}_{j_1 j_2} - w^{\sigma}_{j_1 j_2}) = \bar{y}^{j_1}_{i_1} \bar{y}^{j_2}_{i_2} (\bar{w}^{\sigma}_{j_1 j_2} - w^{\sigma}_{j_1 j_2}) = 0$, which implies, again using regularity of the matrix $\bar{y}^{j}_{i}$, that $\bar{w}^{\sigma}_{j_1 j_2} = w^{\sigma}_{j_1 j_2}$.

Now assume that $\bar{w}^{\sigma}_{j_1 j_2 \cdots j_k} = w^{\sigma}_{j_1 j_2 \cdots j_k}$ for all $k \leq s - 1$. Then (3.11) reduces to

$$\bar{y}^{j_1}_{i_1} \bar{y}^{j_2}_{i_2} \cdots \bar{y}^{j_s}_{i_s} (\bar{w}^{\sigma}_{j_1 j_2 \cdots j_s} - w^{\sigma}_{j_1 j_2 \cdots j_s}) = 0,$$

which gives us $\bar{w}^{\sigma}_{j_1 j_2 \cdots j_s} - w^{\sigma}_{j_1 j_2 \cdots j_s} = 0$ as required.

This completes the proof. □

Denote

$$\Delta_i = z_i^s d_s. \tag{3.12}$$

Properties of the group action of L_n^r on Imm $T_n^r Y$ can now be summarized as follows.

Corollary 3. *The group action* (3.1) *is expressed on* W *by the equations*

$$\bar{y}^k = y^k, \qquad \bar{y}^k_{i_1 i_2 \cdots i_s} = \sum_{p=1}^{s} \sum_{(I_1, I_2, \ldots, I_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_p}_{I_p} y^k_{j_1 j_2 \cdots j_p},$$

$$\bar{w}^\sigma_{j_1 j_2 \cdots j_s} = w^\sigma_{j_1 j_2 \cdots j_s}, \quad 0 \le s \le r.$$

Equations $w^\sigma_{j_1 j_2 \cdots j_s} = c^\sigma_{j_1 j_2 \cdots j_s}$, *where* $c^\sigma_{j_1 j_2 \cdots j_s} \in \mathbb{R}$ *are equations of the orbits of this action, and the functions* y^i, $w^\sigma_{j_1 j_2 \cdots j_s}$ *represent a complete system of real-valued* L_n^r*-invariants on* W. *Moreover, each of these invariants arises by applying a sequence of the vector fields* Δ_i *to the invariants* w^σ.

Our aim now will be to express the vector fields Δ_i in terms of the adapted charts (W, Φ) (Theorem 2).

Corollary 4. *The vector field* Δ_i *has an expression*

$$\Delta_i(J_0^r \zeta) = \frac{\partial}{\partial y^i} + \sum_{l=0}^{r-1} \sum_{p_1 \le p_2 \le \cdots \le p_l} w^\nu_{p_1 p_2 \cdots p_l i}(J_0^r \zeta) \left(\frac{\partial}{\partial w^\nu_{p_1 p_2 \cdots p_l}} \right)_{J_0^r \zeta}$$
$$+ \sum_{l=1}^{r-1} \sum_{p_1 \le p_2 \le \cdots \le p_l} z_i^s (J_0^r \zeta) y^k_{p_1 p_2 \cdots p_l s}(J_0^r \zeta) \left(\frac{\partial}{\partial y^k_{p_1 p_2 \cdots p_l}} \right)_{J_0^r \zeta}. \tag{3.13}$$

Proof. We proceed by direct computation, using (2.5) and Theorem 2. □

Note that at every point of its domain, the vector fields Δ_i (3.12) span an n-dimensional vector subspace of the tangent space of Imm $T_n^{r-1} Y$, determined independently of charts. Indeed, if (V, ψ), and $(\bar{V}, \bar{\psi})$ are two charts, then by (2.6),

$$\bar{\Delta}_i = \bar{z}_i^s \bar{d}_s = \bar{z}_i^s d_s = \bar{z}_i^s \delta_s^p d_p = \bar{z}_i^s y_s^q z_q^p d_p = \bar{z}_i^s y_s^q \Delta_p. \tag{3.14}$$

In the following corollary we use these vector fields to derive the transformation properties of the functions $w^\nu_{p_1 p_2 \cdots p_k}$. Denote $P = (P_j^i)$, where

$$P_j^i = \frac{\partial \bar{y}^i}{\partial y^j} + w_j^\nu \frac{\partial \bar{y}^i}{\partial w^\nu}.$$

Taking $r = 1$ in (3.14), we get

$$\bar{\Delta}_i = \frac{\partial}{\partial \bar{y}^i} + \bar{w}_i^\nu \frac{\partial}{\partial \bar{w}^\nu} = \bar{z}_i^s y_s^q \Delta_q$$
$$= \bar{z}_i^s y_s^q \left(\frac{\partial \bar{y}^j}{\partial y^q} + w_j^\nu \frac{\partial \bar{y}^j}{\partial w^\nu} \right) \frac{\partial}{\partial \bar{y}^j} + \bar{z}_i^s y_s^q \left(\frac{\partial \bar{w}^\nu}{\partial y^q} + w_q^\lambda \frac{\partial \bar{w}^\nu}{\partial w^\lambda} \right) \frac{\partial}{\partial \bar{w}^\nu},$$

from which it follows that $\delta_i^j = \bar{z}_i^s y_s^q P_i^j$, $\bar{w}_i^\nu = \bar{z}_i^s y_s^q \Delta_q \bar{w}^\nu$. The first of these conditions implies that the matrix P is regular, and its inverse, $P^{-1} = Q = (Q_j^i)$, satisfies

$$Q_j^i = \bar{z}_j^s y_s^i.$$

From the second condition we derive the following formula $\bar{w}_i^\nu = Q_i^q \Delta_q \bar{w}^\nu$.

Corollary 5. *Let* (V, ψ), $\psi = (y^A)$ *and* $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^A)$ *be two charts on* Y *such that* $V \cap \bar{V} \neq \emptyset$. *Consider the associated charts* (V_n^r, ψ_n^r) *and* $(\bar{V}_n^r, \bar{\psi}_n^r)$ *and the charts* (W, Φ) *and* $(\bar{W}, \bar{\Phi})$ *on* $\operatorname{Imm} T_n^r Y$. *Let the transformation equations from* (V, ψ) *to* $(\bar{V}, \bar{\psi})$ *be written in the form*

$$\bar{y}^i = F^i(y^k, w^\nu), \quad \bar{w}^\sigma = F^\sigma(y^k, w^\nu).$$

Then the functions $w^\nu_{i_1 i_2 \cdots i_k i_{k+1}}$ *obey the transformation formulas*

$$\bar{w}^\nu_{i_1 i_2 \cdots i_k i_{k+1}} = Q^p_{i_{k+1}} \Delta_p \bar{w}^\nu_{i_1 i_2 \cdots i_k}. \tag{3.15}$$

Proof. By hypothesis, $\det(y_i^k) \neq 0$, hence $\det(\bar{y}_i^k) \neq 0$. Therefore, using (3.7) we get $\bar{w}^\nu_{i_1 i_2 \cdots i_k i_{k-1}} = \bar{z}^s_{i_{k+1}} \delta_s^j d_j \bar{w}^\nu_{i_1 i_2 \cdots i_k} = \bar{z}^s_{i_{k+1}} y_s^p z_p^j d_j \bar{w}^\nu_{i_1 i_2 \cdots i_k} = Q^p_{i_{k+1}} \Delta_p \bar{w}^\nu_{i_1 i_2 \cdots i_k}$. □

A point of $P_n^r Y$ containing a regular (r, n)-velocity $J_0^r \zeta$ is called an (r, n)-*contact element*, or an r-*contact element* of an n-dimensional submanifold of Y, and is denoted by $[J_0^r \zeta]$. As in the case of r-jets, the point $0 \in \mathbb{R}^n$ (resp. $\zeta(0) \in Y$) is called the *source* (resp. the *target*) of $[J_0^r \zeta]$. The set G_n^r of (r, n)-contact elements with source $0 \in \mathbb{R}^n$ and target $0 \in \mathbb{R}^{n+m}$, endowed with the natural smooth structure, is called the (r, n)-*Grassmannian*, or simply a *higher-order Grassmannian*. It is standard to check that the manifold $P_n^r Y = \operatorname{Imm} T_n^r Y / L_n^r$ is a fiber bundle over Y with fiber G_n^r. $P_n^r Y$ with this structure is called the (r, n)-*Grassmannian bundle*, or simply a *higher-order Grassmannian bundle* over Y.

Besides the quotient projection $\rho_n^r : \operatorname{Imm} T_n^r Y \to P_n^r$ (Corollary 1) we have for every s, $0 \leq s \leq r$, the *canonical projection* of $P_n^r Y$ onto $P_n^s Y$ defined by $\rho_n^{r,s}([J_0^r \zeta]) = [J_0^s \zeta]$.

Now we are going to introduce some charts on the manifold of contact elements $P_n^r Y$. To this purpose we consider the *adapted charts* on $\operatorname{Imm} T_n^r Y$, (W, Φ),

$$\Phi = (w^\sigma, w^\sigma_{p_1}, w^\sigma_{p_1 p_2}, \ldots, w^\sigma_{p_1 p_2 \cdots p_r}, y^i, y^i_{j_1}, y^i_{j_1 j_2}, \ldots, y^i_{j_1 j_2 \cdots j_r}),$$

introduced in Theorem 2. We denote $\tilde{W} = \rho_n^r(W)$, and if $J_n^r \zeta \in W$, we define

$$\tilde{\Phi} = (\tilde{y}^i, \tilde{w}^\sigma, \tilde{w}^\sigma_{j_1}, \tilde{w}^\sigma_{j_1 j_2}, \ldots, \tilde{w}^\sigma_{j_1 j_2 \cdots j_r})$$

by

$$\tilde{y}^i([J_0^r \zeta]) = y^i(J_0^r \zeta), \qquad \tilde{w}^\sigma_{j_1 j_2 \ldots j_k}([J_0^r \zeta]) = w^\sigma_{j_1 j_2 \ldots j_k}(J_0^r \zeta). \tag{3.16}$$

Then the pair $(\tilde{W}, \tilde{\Phi})$ is the *associated chart* on $P_n^r Y$. In terms of (W, Φ) and $(\tilde{W}, \tilde{\Phi})$ the quotient projection ρ_n^r is expressed by the equations

$$\tilde{y}^i \circ \rho_n^r = y^i, \qquad \tilde{w}^\sigma_{j_1 j_2 \cdots j_k} \circ \rho_n^r = w^\sigma_{j_1 j_2 \cdots j_k}. \tag{3.17}$$

Consider a point $J_0^r\zeta \in W$, and the vector subspace of the tangent space $T_{\rho_n^r(J_0^r\zeta)}P_n^rY$ spanned by the vectors $T_{J_0^r\zeta}\rho_n^r \cdot \Delta_i(J_0^r\zeta)$, where the vectors $\Delta_i(J_0^r\zeta)$ are defined by (3.13) and (2.5). Indeed, this vector subspace is independent of the choice of a chart used in the definition of d_i. It follows from (3.13) and (3.17) that the vector field Δ_i is ρ_n^r-projectable, and its ρ_n^r-projection is the vector field

$$\tilde{\Delta}_i = \frac{\partial}{\partial \tilde{y}^i} + \sum_{p=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_p} \tilde{w}^\sigma_{j_1 j_2 \cdots j_p i} \frac{\partial}{\partial \tilde{w}^\sigma_{j_1 j_2 \cdots j_p}}.$$

Thus, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Imm}\, T_n^r & \xrightarrow{\Delta_i} & T\mathrm{Imm}\, T_n^{r-1} \\ \downarrow \rho_n^r & & \downarrow T\rho_n^r \\ P_n^r Y & \xrightarrow{\tilde{\Delta}_i} & TP_n^{r-1}Y \end{array}$$

From now on we adopt the standard convention for writing fibered coordinates, and we omit the tilde over the coordinate functions on the left in (3.16). Then the coordinate functions of the chart $(\tilde{W}, \tilde{\Phi})$ will be denoted simply by $\tilde{\Phi} = (y^i, w^\sigma, w^\sigma_{j_1}, w^\sigma_{j_1 j_2}, \ldots, w^\sigma_{j_1 j_2 \cdots j_r})$.

Let us consider two charts (V, ψ), $\psi = (y^A)$, and $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^A)$, such that $V \cap \bar{V} \neq \emptyset$, and the associated charts (V_n^r, ψ_n^r) and $(\bar{V}_n^r, \bar{\psi}_n^r)$ on $\mathrm{Imm}\, T_n^rY$. The transformation equations for the corresponding associated charts on P_n^rY are given by $\bar{w}^\nu_{i_1 i_2 \cdots i_k i_{k+1}} = Q^p_{i_{k+1}} \Delta_p \bar{w}^\nu_{i_1 i_2 \cdots i_k}$ (Eq. (3.15)).

4. Scalar invariants of (r, n)-velocities

Our aim in this section will be to describe all continuous L_n^r-invariant, real-valued functions on the manifold of (r, n)-velocities T_n^rY.

As in the case of regular (r, n)-velocities, we denote by $\rho_n^r : T_n^rY \to T_n^rY/L_n^r$ the *canonical quotient projection*. The quotient set T_n^rY/L_n^r will be considered with its *canonical topological structure*; then ρ_n^r is an *open* mapping. The set $\mathrm{Imm}\, T_n^rY$ is an open, dense, subset of T_n^rY. We have the canonical projection $\pi_n^r : T_n^rY/L_n^r \to Y$, as well as its restriction $\pi_n^r : \mathrm{Imm}\, T_n^rY/L_n^r \to Y$ to the *(r, n)-Grassmann bundle* $P_n^rY = \mathrm{Imm}\, T_n^rY/L_n^r$, which are both continuous. These mappings define a commutative diagram

$$\begin{array}{ccc} \mathrm{Imm}\, T_n^rY & \to & T_n^{r-1}Y \\ \downarrow & & \downarrow \\ P_n^rY & \to & T_n^rY/L_n^r \\ \downarrow & & \downarrow \\ Y & \to & Y \end{array}$$

P_n^rY is an open, dense subset of T_n^rY/L_n^r. Indeed P_n^rY is open in T_n^rY/L_n^r by the definition of the quotient topology, since $\mathrm{Imm}\, T_n^rY = (\rho_n^r)^{-1}(P_n^rY)$ is open in T_n^rY. If $[J_0^r\chi_0] \in T_n^rY/L_n^r$ is such that $[J_0^r\chi_0] \notin P_n^rY$, and W is a neighborhood of $[J_0^r\chi_0]$, then $(\rho_n^r)^{-1}(W)$

is an open set in $T_n^r Y$ containing $[J_0^r \chi_0]$ as a subset. Since Imm $T_n^r Y$ is dense in $T_n^r Y$, $(\rho_n^r)^{-1}(W) \cap \mathrm{Imm}\, T_n^r Y$ is a nonempty open subset of Imm $T_n^r Y$, and since ρ_n^r is open, the set $\rho_n^r((\rho_n^r)^{-1}(W) \cap \mathrm{Imm}\, T_n^r Y)$ is open in $P_n^r Y$. But $\rho_n^r((\rho_n^r)^{-1}(W) \cap \mathrm{Imm}\, T_n^r Y) \subset W$ which means that W contains an element of the set $P_n^r Y$.

Any continuous function on a subset of $P_n^r Y$ defines, when composed with the quotient projection $\rho_n^r : \mathrm{Imm}\, T_n^r Y \to P_n^r Y$, an L_n^r-invariant, continuous function on the corresponding subset of Imm $T_n^r Y$, and vice versa, any L_n^r-invariant, continuous function on an open, L_n^r-invariant subset of $P_n^r Y$ can be factored through ρ_n^r. Since the values of a continuous, real-valued function on $T_n^r Y / L_n^r$ are uniquely determined by its values on $P_n^r Y$, the projection ρ_n^r is the *basis of L_n^r-invariant functions* on $T_n^r Y$.

It is now clear that our problem of finding all continuous L_n^r-invariant, real-valued function on $T_n^r Y$ is equivalent with the problem of finding continuous functions on open subset of the quotient $T_n^r Y / L_n^r$. This gives rise to the problem of *continuous prolongation* of functions on $P_n^r Y$ to the quotient space $T_n^r Y / L_n^r$.

First we need to discuss separability of the points on $T_n^r Y / L_n^r$. It is easily seen that the quotient topology on $T_n^r Y / L_n^r$ is *not* Hausdorff.

Note that any two points $[J_0^r \chi_0], [J_0^r \chi] \in T_n^r Y / L_n^r$ such that $\chi_0(0) \neq \chi(0)$, can always be separated by open sets. This follows from the continuity of the quotient projection of π_n^r, and from separability of Y. To study the situation in the fibers, we prove the following lemma.

Lemma 4. *Let $y \in Y$ be a point, (V, ψ), $\psi = (y^A)$ a chart at y, and $J_0^r \chi_0 \in (\tau_n^{r,0})^{-1}(y)$ the (r, n)-velocity with target y defined by $J_0^r \chi_0 = (y^A, 0, 0, \ldots, 0)$ in the associated chart. Then any L_n^r-invariant neighborhood of $J_0^r \chi_0$ contains the fiber $(\tau_n^{r,0})^{-1}(y)$.*

Proof. The fiber $(\tau_n^{r,0})^{-1}(y) = (\rho_n^{r,0})^{-1}((\pi_n^{r,0})^{-1}(y))$ over $y \in Y$ in $T_n^r Y$ is endowed with the induced chart (V_n^r, ψ_n^r), $\psi_n^r = (y^A, y_{j_1}^A, y_{j_1 j_2}^A, \ldots, y_{j_1 j_2 \cdots j_r}^A)$. The coordinates $\bar{y}^A, \bar{y}_{j_1 j_2 \cdots j_s}^A$ of the points of the orbit $[J_0^r \chi_0]$ are given by (2.16),

$$\bar{y}^A = y^A, \qquad \bar{y}_{j_1 j_2 \cdots j_s}^A = \sum_{p=1}^{s} \sum_{(I_1, I_2, \ldots, I_p)} a_{I_1}^{j_1} a_{I_2}^{j_2} \cdots a_{I_p}^{j_p} y_{j_1 j_2 \cdots j_p}^A,$$

where $J_0^r \alpha \in L_n^r$, $J_0^r \alpha = (a_{j_1}^i, a_{j_1 j_2}^i, \ldots, a_{j_1 j_2 \cdots j_r}^i)$. Thus, $\bar{y}^A = y^A$, $\bar{y}_{j_1 j_2 \ldots j_s}^A = 0$, which means that the orbit $[J_0^r \chi_0]$ consists of a single point. Let W be an L_n^r-invariant neighborhood of the point $J_0^r \chi_0$. We show that each orbit in $(\tau_n^{r,0})^{-1}(y)$ has a nonempty intersection with W. Then we apply L_n^r-invariance to obtain the inclusion $(\tau_n^{r,0})^{-1}(y) \subset W$.

Let us consider an arbitrary element $J_0^r \chi = (y^A, y_{j_1}^A, y_{j_1 j_2}^A, \ldots, y_{j_1 j_2 \cdots j_r}^A) \in (\tau_n^{r,0})^{-1}(y)$, and a one-parameter family of velocities $J_0^r \chi \circ J_0^r \beta_\tau$ in $(\tau_n^{r,0})^{-1}(y)$ defined in components by

$$\beta_\tau = (\beta_\tau^i), \ \beta_\tau^i(t^1, t^2, \ldots, t^n) = \tau t^i,$$

where $0 \leq \tau \leq 1$. Then $J_0^r \beta_\tau = (\tau \delta_j^i, 0, 0, \ldots, 0)$, and by (2.16), the L_n^r-orbit of $J_0^r \chi$ contains the points $J_0^r \chi \circ J_0^r \beta_\tau = (\bar{y}^A, \bar{y}_{j_1}^A, \bar{y}_{j_1 j_2}^A, \ldots, \bar{y}_{j_1 j_2 \cdots j_r}^A)$ given by $\bar{y}^A = y^A$, $\bar{y}_{i_1 i_2 \cdots i_s}^A = \tau^s \bar{y}_{i_1 i_2 \cdots i_s}^A$. Clearly, for sufficiently small τ, $J_0^r \chi \circ J_0^r \beta_\tau \in W$.

This shows that the orbits passing through any neighborhood of the point $J_0^r \chi_0$, where $\chi_0(0) = y$, fill the whole fiber $(\tau_n^{r,0})^{-1}(y)$. □

Consider a point $y \in Y$, a chart (V, ψ), $\psi = (y^A)$ at y, and the L_n^r-orbit $[J_0^r \chi_0]$ of the velocity $J_0^r \chi_0 = (y^A, 0, 0, \ldots, 0) \in (\tau_n^{r,0})^{-1}(y)$. Lemma 4 shows that any neighborhood of the orbit $[J_0^r \chi_0] \in T_n^r Y/L_n^r$ contains the fiber $(\pi_n^r)^{-1}(y)$ in $T_n^r Y/L_n^r$ over y. This proves, in particular, that no point of $(\pi_n^r)^{-1}(y)$ can be separated from $[J_0^r \chi_0]$ by open sets.

This gives us the following theorem saying that if a continuous invariant is defined on a fiber in $T_n^r Y$, then it is constant along this fiber.

Theorem 3. *Let W be an open, L_n^r-invariant set in* Imm $T_n^r Y$, $f : W \to \mathbb{R}$ *an L_n^r-invariant function. Assume that W contains two regular velocities $J_0^r \zeta$, $J_0^r \chi$ with common target $y = \zeta(0) = \chi(0)$ such that $f(J_0^r \zeta) \neq f(J_0^r \chi)$. Then f cannot be continuously prolonged to the fiber* $(\tau_n^{r,0})^{-1}(y) \subset$ Imm $T_n^r Y$.

Proof. Indeed, since $\mathbb{R}$ is Hausdorff, any continuous, L_n^r-invariant, real-valued function takes the same value at the points which cannot be separated by open sets. Assume that f can be prolonged to the fiber $(\tau_n^{r,0})^{-1}(y) \subset$ Imm $T_n^r Y$. Then by Lemma 4, f is equal along the fiber $\tau_n^{r,0}(y)$ to $f(J_0^r \chi_0) =$ const, which is a contradiction. □

In particular, none of the L_n^r-invariant functions $w^{\sigma}_{j_1 j_2 \cdots j_k}$ (Theorem 2) can be prolonged to a fiber $(\tau_n^{r,0})^{-1}(y)$.

Now it is immediate that each L_n^r-invariant function on $T_n^r Y$ is trivial in the following sense.

Corollary 6. *A, continuous function $f : T_n^r Y \to \mathbb{R}$ is L_n^r-invariant if and only if $f = F \circ \tau_n^{r,0}$, where $F : Y \to \mathbb{R}$ is a continuous function.*

Appendix A. Regular $(2, n)$-velocities

As before, Y denotes a smooth manifold of dimension $n + m$. In this section we consider the manifold Imm $T_n^2 Y$ of regular $(2, n)$-velocities on Y, and the Grassmann bundle $P_n^2 Y$. We wish to collect in an explicit form all basic formulas concerning charts and invariants in this case, which will be important for applications.

If (V, ψ), $\psi = (y^A)$, is a chart on Y, define $V_n^2 = (\tau_n^{2,0})^{-1}(V)$ and $\psi_n^2 = (y^A, y_i^A, y_{ij}^A)$ where $1 \leq A \leq n + m$, $1 \leq i \leq j \leq n$, by the formulas $y^A(J_0^2 \zeta) = y^A(\zeta(0))$, $y_i^A(J_0^2 \zeta) = D_i y^A(\zeta)(0)$, $y_{ij}^A(J_0^2 \zeta) = D_i D_j (y^A \zeta)(0)$. If $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^A)$, is another chart on Y, and the transformation equations are written as $\bar{y}^A = F^A(y^B)$, then

$$\bar{y}^A = F^A(y^B), \quad \bar{y}_i^A = \frac{\partial F^A}{\partial y^B} y_i^B, \quad \bar{y}_{ij}^A = \frac{\partial^2 F^A}{\partial Y^B \partial y^C} y_i^B y_j^C + \frac{\partial F^A}{\partial y^B} y_{ij}^B \tag{A.1}$$

on $V_n^2 \cap \bar{V}_n^2$ (see (2.9)–(2.11)). If $f : V_n^1 \to \mathbb{R}$ is a smooth function, we define a function $d_i f : V_n^2 \to \mathbb{R}$ by

$$d_i f = \frac{\partial f}{\partial y^A} y_i^A + \frac{\partial f}{\partial y_j^A} y_{ij}^A.$$

This function is called the ith *formal derivative* of f. In particular, $d_i y^A = y_i^A$, $d_i y_j^A = y_{ij}^A$.

By definition, $\operatorname{rank}(y_s^B(J_0^2\zeta)) = n$ at every point $J_0^2\zeta \in V_n^2$. Thus, there exists a subsequence $I = (A_1, A_2, \dots, A_n)$ of the sequence $(1, 2, \dots, n, n+1, n+m)$ such that $\det(y_i^{A_i}(J_0^2\zeta)) \neq 0$. Denote $V_n^{2(I)} = \{J_0^2\zeta \in V_n^2 \mid \det(y_j^{A_i}(J_0^2\zeta)) \neq 0\}$. If $\psi_n^{2(I)}$ is the restriction of ψ_n^2 to $V_n^{2(I)}$, then the pair $(V_n^{2(I)}, \psi_n^{2(I)})$, $\psi_n^{2(I)} = (y^A, y_i^A, y_{ij}^A)$, is a chart on $\operatorname{Imm} T_n^2 Y$, and

$$\bigcup_I V_n^{2(I)} = V_n^2.$$

By (2.14), the group multiplication $(J_0^2\alpha, J_0^2\beta) \to J_0^r\alpha \circ J_0^r\beta$ in the second differential group L_n^2 of $\mathbb{R}^n$ is given in the *canonical coordinates* by

$$c_i^k = b_i^p a_p^k, \qquad c_{ij}^k = b_{ij}^p a_p^k + b_i^p b_j^q a_{pq}^k. \tag{A.2}$$

Indeed, in this formula a_p^k, a_{pq}^k (resp. b_i^p, b_{ij}^p, resp. c_i^k, c_{ij}^k) are the coordinates of $J_0^2\alpha$ (resp. $J_0^r\beta$, resp. $J_0^r\alpha \circ J_0^r\beta$). L_n^2 acts on $\operatorname{Imm} T_n^2 Y$ smoothly to the right by the jet composition $(J_0^2\zeta \circ J_0^2\alpha) \to J_0^2\zeta \circ J_0^2\alpha$. This action is expressed by

$$\bar{y}^A = y^A, \qquad \bar{y}_i^A = y_s^A a_i^s, \qquad \bar{y}_{ij}^A = y_{pq}^A a_i^p a_j^q + y_p^A a_{ij}^p, \tag{A.3}$$

where y^A, y_i^A, y_{ij}^A are the coordinates of a velocity $J_0^2\zeta$, and $\bar{y}^A$, $\bar{y}_i^A$, $\bar{y}_{ij}^A$ are the coordinates of the transformed velocity $J_0^2\zeta \circ J_0^2\alpha$.

Now we are going to construct an atlas on $\operatorname{Imm} T_n^2 Y$, adapted to this group action. Given a chart (V, ψ), $\psi = (y^A)$, we note that the action (A.3) preserves each of the sets $V_n^{2(I)}$. Indeed, if $J_0^2\zeta \in V_n^{2(I)}$, then by definition, the matrix $y_j^{A_i} = y_j^{A_i}(J_0^2\zeta)$ is of maximal rank, and the second equation of (A.3) implies that the matrix $\bar{y}_j^{A_i} = y_j^{A_i}(J_0^2(\zeta\alpha))$ of the transformed point is also of maximal rank.

Consider for example the case $I = (1, 2, \dots, n)$. Then $\det(y_j^i) \neq 0$ for $1 \leq i, j \leq n$ (i.e. on $V_n^{2(I)}$). We define smooth functions $z_j^i : V_n^{2(I)} \to \mathbb{R}$ by $z_p^i y_j^p = \delta_j^i$. These functions form a regular matrix on $V_n^{2(I)}$. Eqs. (A.3) then give for $A = k = 1, 2, \dots, n$

$$z_p^k \bar{y}_i^p = a_i^k, \qquad z_s^k \bar{y}_{ij}^s = z_s^k y_{pq}^s a_i^p a_j^q + z_s^k y_p^s a_{ij}^p = z_s^k y_{pq}^s z_r^p \bar{y}_i^r z_t^q \bar{y}_j^t + a_{ij}^k,$$

and for $A = \sigma = n+1, n+2, \dots, m$

$$\bar{y}^A = y^A, \qquad \bar{y}_i^A = y_s^A z_p^s \bar{y}_i^p,$$
$$\bar{y}_{ij}^A = y_{pq}^A z_s^p \bar{y}_i^s z_t^q \bar{y}_j^t + y_k^A (z_s^k \bar{y}_{ij}^s - z_s^k y_{pq}^s z_r^p \bar{y}_i^r z_t^q \bar{y}_j^t).$$

Since the second formula gives us the relation $\bar{z}^i_j \bar{y}^\sigma_i = z^s_j y^\sigma_s$, and the third one implies

$$\bar{z}^i_u \bar{z}^j_v (\bar{y}^\sigma_{ij} - \bar{z}^k_s \bar{y}^\sigma_k \bar{y}^s_{ij}) = y^\sigma_{pq} z^p_u z^q_v - z^k_s y^\sigma_k y^s_{pq} z^p_u z^q_v = z^p_u z^q_v (y^\sigma_{pq} - z^k_s y^\sigma_k y^s_{pq}),$$

we finally get

$$\bar{y}^A = y^A, \qquad \bar{z}^i_j \bar{y}^\sigma_i = z^s_j y^\sigma_s,$$

and $\bar{z}^i_u \bar{z}^j_v (\bar{y}^\sigma_{ij} - \bar{z}^k_s \bar{y}^\sigma_{pq} \bar{y}^s_{ij}) = z^p_u z^q_v (y^\sigma_{pq} - z^k_s y^\sigma_k y^s_{pq})$. Therefore, the functions

$$y^i, \quad w^\sigma = y^\sigma, \quad w^\sigma_i = z^s_i y^\sigma_s, \quad w^\sigma_{ij} = z^p_i z^q_j (y^\sigma_{pq} - z^k_s y^\sigma_k y^s_{pq}) \tag{A.4}$$

are *constant* along the L^2_n-orbits in $V^{2(I)}_n \subset \operatorname{Imm} T^2_n Y$, and the functions y^i, w^σ, w^σ_i, w^σ_{ij}, y^i_j, y^i_{jk} define a new chart on the set $V^{2(I)}_n$ *adapted* to the group action of L^2_n. The right action (A.3) is expressed in terms of this new chart by $\bar{y}^i = y^i$, $\bar{w}^\sigma = w^\sigma$, $\bar{w}^\sigma_{ij} = w^\sigma_{ij}$, $\bar{y}^k_i = y^k_s a^s_i$, $\bar{y}^k_{ij} = y^k_{pq} a^p_i a^q_j + y^k_p a^p_{ij}$. The functions (A.4) are components of the quotient projection ρ^2_n of the manifold of regular $(2, n)$-velocities $\operatorname{Imm} T^2_n Y$ onto the $(2, n)$-Grassmann bundle $P^2_n Y$, and form a *basis of L^2_n-invariant functions* on $V^{2(I)}_n$.

A direct interpretation of the coordinate functions (A.4) is obtained as follows. Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a chart on Y, and let $J^2_0\zeta \in V^{2(I)}_n$. We assign to the 2-jet $J^2_0\zeta$ an element $J^2_0\alpha \in L^2_n$ by means of a representative α satisfying, in addition to the condition $\alpha(0) = 0$, the following two conditions $a^s_i(J^2_0\alpha) = D_i\alpha^s(0) = y^s_i(J^2_0\zeta)$, $a^s_{ij}(J^2_0\alpha) = D_i D_j \alpha^s(0) = y^s_{ij}(J^2_0\zeta)$. Then it is easily seen that

$$a^i_r(J^2_0\alpha^{-1}) = D_r(\alpha^{-1})^i(0) = z^i_r(J^2_0\zeta),$$
$$a^i_{rs}(J^2_0\alpha^{-1}) = D_r D_s(\alpha^{-1})^i(0) = -z^j_r(J^2_0\zeta) z^k_s a^i_r(J^2_0\zeta) y^p_{jk}(J^2_0\zeta),$$

and we get for the coordinates of the 2-jet $J^2_0(\zeta \circ \alpha^{-1}) \in V^{2(I)}_n$, by (A.4),

$$y^i(J^2_0(\zeta \circ \alpha^{-1})) = y^i(J^2_0\zeta), \quad y^k_i(J^2_0(\zeta \circ \alpha^{-1})) = \delta^k_i, \quad y^k_{ij}(J^2_0(\zeta \circ \alpha^{-1})) = 0,$$
$$y^\sigma(J^2_0(\zeta \circ \alpha^{-1})) = w^\sigma(J^2_0\zeta), \quad y^\sigma_i(J^2_0(\zeta \circ \alpha^{-1})) = w^\sigma_i(J^2_0\zeta),$$
$$y^\sigma_{ij}(J^2_0(\zeta \circ \alpha^{-1})) = w^\sigma_{ij}(J^2_0\zeta),$$

where $k = 1, 2, \ldots, n \quad \sigma = n+1, n+2, \ldots, m$. This represents the desired interpretation of the functions (A.4) as jet coordinates of the 2-jets $J^2_0\zeta \circ J^2_0\alpha$, with $J^2_0\alpha$ determined by the considered chart.

One can determine the transformation formulas from $\psi^2_n(J^2_0(\zeta \circ \alpha^{-1}))$ to $\bar{\psi}^2_n(J^2_0(\zeta \circ \bar{\alpha}^{-1}))$, with obvious meaning of $\bar{\alpha}$. These formulas illustrate the well-known fact that the higher-order Grassmann bundles have a relatively complicated smooth structure. Consider an element $J^2_0\zeta \in V^{2(I)}_n \cap \bar{V}^{2(I)}_n$. The corresponding computations for $J^2_0\zeta \in V^{2(J)}_n \cap \bar{V}^{2(J)}_n$ with arbitrary I, J are quite analogous. We have

$$\begin{aligned}\bar{\psi}^2_n(J^2_0(\zeta \circ \bar{\alpha}^{-1})) &= \bar{\psi}^2_n(J^2_0(\zeta \circ \alpha^{-1} \circ \alpha\bar{\alpha}^{-1})) = \bar{\psi}^2_n(J^2_0(\zeta \circ \alpha^{-1})) \circ J^2_0(\alpha\bar{\alpha}^{-1})) \\ &= \bar{\psi}^2_n(\psi^2_n)^{-1}(\psi^2_n(J^2_0(\zeta \circ \alpha^{-1}) \circ J^2_0(\alpha\bar{\alpha}^{-1}))).\end{aligned}$$

To derive explicit expressions, one has to substitute for the group multiplication (A.2), the group action (A.3), and the transformation (A.1) in this formula. Denoting

$$P_s^q = \frac{\partial F^q}{\partial y^s} + \frac{\partial F^q}{\partial y^\nu} w_s^\nu,$$

we get a regular matrix $P = (P_s^q)$. Let $Q = P^{-1} = (Q_j^i)$ be its inverse. Then

$$\bar{y}_i^q = P_s^q y_i^s, \quad \bar{z}_i^q = Q_i^s z_s^q.$$

After a tedious but straightforward calculation we get the following result. Given the transformation equations on Y, $\bar{y}^k = F^k(y^p, y^\nu)$, $\bar{y}^\sigma = F^\sigma(y^p, y^\nu)$, then on $V_n^{2(I)} \cap \bar{V}_n^{2(J)}$,

$$\bar{y}^k = F^k(y^p, y^\nu), \quad \bar{w}^\sigma = F^\sigma(y^p, y^\nu), \quad \bar{w}_i^\sigma = Q_i^p \left(\frac{\partial F^\sigma}{\partial y^p} + \frac{\partial F^\sigma}{\partial w^\nu} w_p^\nu \right),$$

$$\begin{aligned} \bar{w}_{ij}^\sigma = {} & Q_i^p Q_j^q \left(\frac{\partial^2 F^\sigma}{\partial y^p \partial y^q} + \frac{\partial^2 F^\sigma}{\partial y^p \partial w^\nu} w_q^\nu + \frac{\partial^2 F^\sigma}{\partial w^\nu \partial y^q} w_p^\nu \right. \\ & \left. + \frac{\partial^2 F^\sigma}{\partial w^\mu \partial w^\nu} w_p^\mu w_q^\nu + \frac{\partial F^\sigma}{\partial w^\nu} w_{pq}^\nu \right) \\ & - Q_i^a Q_j^b Q_k^p \left(\frac{\partial F^\sigma}{\partial y^p} + \frac{\partial F^\sigma}{\partial w^\nu} w_p^\nu \right) \\ & \times \left(\frac{\partial^2 F^k}{\partial y^a \partial y^b} + \frac{\partial^2 F^k}{\partial y^a \partial w^\sigma} w_b^\sigma + \frac{\partial^2 F^k}{\partial w^\sigma \partial y^b} w_a^\sigma \right. \\ & \left. + \frac{\partial^2 F^k}{\partial w^\sigma \partial w^\lambda} w_a^\sigma w_b^\lambda + \frac{\partial F^k}{\partial w^\lambda} w_{ab}^\lambda \right). \end{aligned}$$

Clearly, these equations represent the transformation formulas for the induced charts on the $(2, n)$-Grassmann bundle $P_n^2 Y$.

References

[1] P. Dedecker, On the generalization of symplectic geometry to multiple integrals in the calculus of variations, Lecture Notes in Mathematics, Vol. 570 (Springer, Berlin) 395–456.

[2] J. Dieudonné, *Elements d'Analyse 3* (Gauthier-Villars, Paris, 1970).

[3] D.R. Grigore, A generalized Lagrangian formalism in particle mechanics and classical field theory, Fortschr. d. Phys. 41 (1993) 569–617.

[4] D.R. Grigore and O. Popp, On the Lagrange–Souriau form in classical field theory, Math. Bohemica, to appear.

[5] P. Horváthy, Variational formalism for spinning particles, JMP 20 (1974) 49–52.

[6] J. Klein, Espaces variationels et Mechanique, Ann. Inst. Fourier (Grenoble) 12 (1962) 1–124.

[7] I. Kolár, P. Michor and J. Slovák, *Natural Operations in Differential Geometry* (Springer, Berlin, 1993).

[8] I.S. Krasilschik, A.M. Vinogradov and V.V. Lychagin, *Geometry of Jet Spaces and Nonlinear Differential Equations* (Gordon and Breach, New York, 1986).

[9] D. Krupka, Local invariants of linear connections, Colloq. Math. Soc. János Bolyai, 31. Differential Geometry (Budapest, 1979) (North-Holland, Amsterdam, 1982) 349–369.

[10] D. Krupka, Natural Lagrangian structures, Semester on Diff. Geom. (Banach Center, Warsaw, 1979) Banach Center Publications 12 (1984) 185–210.

[11] M. Krupka, Orientability of higher order Grassmannians, Math. Slovaca 44 (1994) 107–115.

[12] M. Krupka, Natural operators on vector fields and vector distributions, Ph.D. Dissertation, Faculty of Science, Masaryk University, Brno, Czech Republic (1995)
[13] D. Krupka and J. Janyška, *Lectures on Differential Invariants* (Brno University, Czech Republic, 1990).
[14] A. Nijehuis, Natural bundles and their general properties, Differential Geometry, in honour of K. Yano (Kinokuniya, Tokyo, 1972) 317–334.
[15] J.M. Souriau, *Structure des Systemes Dynamiques* (Dunod, Paris, 1970).